Updated: Feb 2022
Course: SPH 3U1
Unit: Mechanics

Lesson 5, 6: Title: Equations of Motion (2 days)

Bellwork: handout or discuss ISP?

Do students need to put units in all of their calculations? On the one hand this is good, on the other, it makes the algebra harder to read.

~~Preliminaries: quiz on graphs of motion~~

Lesson:

Equations are normally more useful in solving problems than using graphs and a lot more accurate. In all of these equations, acceleration is constant (it may be zero).

We already know two important equations (**definitions**):

$$v = \frac{\Delta d}{\Delta t} \qquad (1) \qquad\qquad \text{and} \qquad a = \frac{\Delta v}{\Delta t}$$

***NOTE**: Equation (1) is only true if a = 0, so we don't use it much since problems where a = 0 are too simple.*

If $a \neq 0$ then it becomes $v_{ave} = \frac{\Delta d}{\Delta t}$ which becomes $\boxed{\frac{v_1 + v_2}{2} = \frac{\Delta d}{t}}$ (2)

One thing that is particularly useful is to write distance and speed as a function of time:

$a = \frac{\Delta v}{\Delta t}$ can be rearranged to get $\boxed{v = at + v_i}$ (3)

Q. What did the graph of velocity vs time look like? A straight line.
You can see that this equation follows the form y = mx + b, so it is a straight line

The distance form of this equation comes from calculus, integrating (3)

$\boxed{d = \frac{1}{2}at^2 + v_i t + d_i}$ or $\boxed{\Delta d = \frac{1}{2}at^2 + v_i t}$ (4)

- *NOTE if a = 0, we get equation (1) v = d/t*
- Q. What did the graph of displacement vs time look like? A parabola.
  Note that this is the equation of a parabola.
- There is always a ½ in it. (This comes from calculus).

One final useful equation: $\boxed{v_f^2 - v_i^2 = 2a\Delta d}$ (5)

We get this one by combining equations (3) and (4) to eliminate time.

- *Use this equation if you don't need to know the time taken.*
- *You can always use (3) and (4) for any problems that would require (5)*
- *This equation doesn't correspond to any graph.*

These equations work as either <u>scalar</u> or <u>vector</u> equations. Note that when you use vectors in equations, you can NEVER mix horizontal and vertical quantities.

NOTE: you can use subscripts f, and i, for final and initial instead of 2 and 1. Try and get used to both types of notation.

Important: know the units that the various quantities have. This will help so much when you read the problem and write down what you know.

t	sec, min, hours
d , $\vec{d}$	m
v, $\vec{v}$	m/s , km/hr
a, $\vec{a}$	m/s^2

There are 5 quantities in these equations: a, v_1, v_2, Δd, t
We're going to ignore the first simple equation v=d/t
The other equations all have 4 quantities and are missing a different one. If you don't know and don't need time, then find the equation that has no t in it: $v_2^2 + v_1^2 = 2a\Delta d$

Wait … if we ignore equation (1) then we only have 4 equations. All of these 4 have v_1 so we need an equation with no v1.
Combining equations (2) and (3) we can get this: $\boxed{\Delta d = -\frac{1}{2}at^2 + v_2 t}$ (6)
We don't use this often as we're normally given v1.

Do these problems on the board:

Emphasize: (i) proper equation solving

(ii) inital and final values

*** Marks are given for: (i) writing down what is given and what you need to find*

(ii) writing down the equation

(iii) solving the problem (using algebra)

(iv) writing the correct answer

(v) correct units on the answer and the 'given'

Important Hints:

1. when an object is dropped $v_1 = 0$
2. when an object is thrown upwards, v_1 cannot be zero; it is positive
3. when an object reaches its maximum height, $v_2 = 0$
   (ie. v at this point in time = 0. We are not saying that v_2 is always zero. v is the variable)
4. When an object returns to the same height from which it was thrown:
   a) $\Delta d = 0$
   b) the total time in the air is twice the time to the maximum height
   c) $v_2 = -v_1$ (the final velocity is equal and opposite to the initial velocity)

1. A ball is thrown upwards at 30 m/s. (*This problem was worked out graphically in a previous lesson.)

(a) How high does it go?

(b) How long is it in the air?

Solution:

$v_1 = 30$ m/s

$a = -10$ m/s^2

$d_1 = 0$

a) **at maximum height $v_2 = 0$**

[method 1]

$$v_2^2 - v_1^2 = 2a\Delta d$$

$0 - (30 \text{ m/s})^2 = 2(-10 \text{ m/s}^2)\Delta d$

$900 \text{ m}^2/\text{s}^2 \div 20 \text{ m/s}^2 = \Delta d$

$\Delta d = 45$ m.

[Method 2]

$$v_2 = at + v_1$$

$0 = (-10 \text{ m/s}^2)\, t + 30 \text{ m/s}$

$t = 30 \text{ m/s} \div 10 \text{ m/s}^2$

$t = 3$ s This is the time to the maximum height

Now put this into equation (4) to find the height

$$d_2 = \tfrac{1}{2}at^2 + v_1 t + d_1$$

$d_2 = ½ (-10 \text{ m/s}^2)(3\text{s})^2 + (30 \text{ m/s})(3\text{s}) + 0$

$d_2 = -45 \text{ m} + 90 \text{ m} = 45$ m

This is the same answer we got with the graphs !

b) We know that when it reaches the ground, the change is position is zero. It is back where it started from.

$$\Delta d = \tfrac{1}{2}at^2 + v_i t$$

$0 = ½ (-10 \text{ m/s}^2)(t)^2 + (30 \text{ m/s})(t)$

$0 = -5\, t^2 + 30\, t$ (divide by –5)

$0 = t^2 - 6t$ (factor)

$0 = t\,(t - 6)$

$\therefore t = 0$ or $t = 6$s

[Method 2]

$$a = \frac{\Delta v}{\Delta t}$$

We know that $v_2 = -v_1$ (from hint above)

$\therefore -10 \text{ m/s}^2 = -30 - 30 / t$

$\therefore t = 6$ sec

We have two answers because the ball is at zero height at zero seconds (before it is thrown) as well as at 6 seconds.

We also could have come up with the answer by noting that it took 3 seconds to reach maximum height, so it will take 3 seconds to come back down (we always ignore air resistance), giving a total of 6 seconds in the air.

2. A rock is dropped into a 100m deep well. How long until the rock hits the bottom?

NOTE: when you drop something the initial velocity is zero.

When we talk about speeds in these problems, it is always right after the object has left your hand, and the instant before it hits the ground. The final velocity of a falling object is never zero.

Note that $d_2 = -100$ m because you are going 100 m below ground level. If you don't have a negative sign you will be taking the √ of a negative number.

Answer: 4.47 s

3. a) A car travels 12 m/s for 3 seconds. What distance does it travel in this time? (Ans. d = 36 m)

b) A car accelerates from 12 m/s to 19 m/s in 3 seconds. What distance did it travel in this time?

YOU CANNOT USE v = d/t BECAUSE V IS CHANGING! $a \neq 0$. Note the difference between a) and b).

This question is easy to do using a graph.

Solution: (i) find acceleration: $a = \Delta v/\Delta t = 7 \text{ m/s} \div 3\text{s} = 2.33 \text{ m/s}^2$

(ii) now find distance using formula (4) or (5):

$$v_2^2 - v_1^2 = 2a\Delta d$$

$$19^2 - 12^2 = 2(2.33)\Delta d$$

$$361 - 144 = 4.66\,\Delta d$$

$\Delta d = 46.5$ m Is this answer reasonable compared to part (a) – yes, we expect a larger answer than 36.

4. You are on a 5m high roof and throw a ball upwards at 10 m/s. It lands on the ground below you.

(a) How long was it in the air? (review quadratic formula).

(b) What is its final velocity? *[what sort of number do you expect for this one? a negative number more than 10 m/s]*

Solution:

$$d = \tfrac{1}{2}at^2 + v_i t + d_i \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$0 = -5t^2 + 10t + 5$ $\qquad t = [2 \pm \sqrt{4-4(1)(-1)}\,] / 2$

$0 = t^2 - 2t - 1$ $\qquad t = [2 \pm \sqrt{8}\,] / 2$

$t = 1 \pm \sqrt{2}$ $\quad \therefore t = -0.414\text{s}$ or $t = 2.414$ s.

Question: what does the negative solution mean? *Sketch the parabola. It is the time taken for the ball to go from the ground up to this position (5 m high); or the time taken to reach the ground if you threw it downwards at –10 m/s!*

(b) $v_2 = at + v_1$

$v_2 = (-10 \text{ m/s}^2)(2.414\text{s}) + 10 \text{ m/s}$

$= -14.14$ m/s

5. An arrow is shot upwards. If it is in the air for 5 seconds, how high did it go?

Solution:

Hint: realize that the time it took to reach the maximum height is 2.5 seconds. Then use $v_2 = at + v_1$ setting $v_2 = 0$. This will give you v_1. Once you have v_1, you can use equation (4) or (5) to find the maximum height.

Types of motion problems:
1. a = 0, use **v=d/t**
2. no distances given / required, use **a = v/t**
3. inital or final velocity zero: If any two of the following 4 variables are known, you can find the other two: a, d, t, other velocity (examples: dropping objects, max height reached)
4. no zero velocities and d1 ≠ d2. Use **quadratic formula** to find t.

Steps for solving motion problems:
1. write down all of the information that you know
2. write down the equation that you are going to use ← you must do this for each problem
   Look for an equation that tells you what you need to know.
3. put numbers in the equation and solve.
   make sure your solutions are neat and logically organized, so they can be followed easily.
4. write a final statement (is this necessary?)

<u>Homework</u>: Read page 44, do p46 #1-5 (this may or may not be assigned)

MORE STUFF ← this is worth doing on the board, Socratically .

Using $d = \frac{1}{2}at^2 + v_i t + d_i$ $\quad\quad \Delta d = \frac{1}{2}at^2 + v_i t$

What situation gives formulas like these:

1. $d = ½ (-10\ m/s^2)(2s)^2 + 0(2s) + 25$
   An object is being dropped from a 25 m cliff. How far has it fallen in the first 2 seconds?

2. $d = ½ (-10\ m/s^2)(2s)^2 + (11\ m/s)(2s) + 0$
   An object is thrown upwards at 11 m/s. How high is it after 2 seconds?

3. $0 – (5m/s)^2 = 2a\ (0.3\ m)$
   A car (or something) slows from 5 m/s to zero in 0.3 metres. What is it's acceleration?

4. $0 = ½ (-10\ m/s^2)(t)^2 + (-4\ m/s)(t) + 25$
   An object is thrown downwards at –4 m/s from the top of a 25 m high cliff. How long until it reaches the ground?

5. $\Delta d = ½ (3\ m/s^2)(1.5s)^2 + 26\ m/s(1.5s)$
   An object is moving at 26 m/s. It accelerates at 3 m/s² for 1.5 seconds. How far has it gone in this time?

6. $0 = ½ (-10\ m/s^2)(4s)^2 + v_1(4s)$
   An object is in the air for 4 seconds (it's total displacement is zero, and we have a –10 m/s² acceleration). What was its initial velocity?

-- any others that you want to add --